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RATIONAL PONTRJAGIN HOMOLOGY RING OF THE BASED LOOP SPACE ON SOME HOMOGENEOUS SPACES

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Dedicated to Professor Mirjana Vuković on the occasion of her 70th birthday

ABSTRACT. In this paper we consider the rational homology of the based loop space on the class of compact, connected homogeneous spaces, known as the generalised symmetric spaces. Due to the results from [11] and [12] together with Milnor-Moore theorem, it is possible to explicitly describe the rational Pontrjagin homology ring of the based loop space on these spaces. By making use of that, we prove some general results on their based loop space rational homology ring and provide an explicit description of the rational Pontrjagin homology ring for some irreducible, simply connected symmetric spaces.

1. INTRODUCTION

A based loop space ΩX of a pointed topological space X is an H - space where one of the possible multiplication is given by loop concatenation. The ring structure in $H_*(\Omega X)$ induced by loop multiplication is called Pontrjagin homology ring.

In this paper we consider the rational Pontrjagin homology ring of the based loop space $\Omega(G/H)$ of the compact connected homogeneous spaces G/H, in particular generalised symmetric spaces. These are defined by the condition that their stationary subgroup H can be obtained as the fixed point subgroup of a finite order automorphism of the group G. These spaces are known [11] to be formal in the sense of the rational homotopy theory. Therefore, one can start with their rational cohomology algebra and apply the methods of the rational homotopy theory together with Milnor and Moore theorem, to compute the rational Pontrjagin homology ring of their based loop spaces. In doing this we appeal to the results from [11], where the rational cohomology algebras for all generalised symmetric spaces with simple compact Lie groups are computed. We also use the results

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from [12], where the previous results are pushed up further and there are described the generators of the minimal models for all generalised symmetric spaces.

By Theorem 4.1 and Theorem 4.2, we first provide the description of the rational Pontrjagin homology ring of the based loop spaces on the generalised symmetric spaces satisfying certain conditions. Moreover, we provide explicit description of the based loop space rational Pontrjagin homology ring of the following irreducible, simply connected symmetric spaces: $SU(2n+1)/SO(2n+1)$, $SO(2n)/Sp(n)$, $SO(2n+1)/SO(2)\times SO(2n-1)$, $SU(2n)/SO(2n)$, $U(n+1)/U(k)\times U(n-k+1)$ and $SO(2n+1)/SO(2k)\times SO(2n+1-2k)$. These computations illustrate the method which can be analogously applied to all other generalised symmetric spaces.

We would like to mention that the same approach from the rational homotopy theory was the starting point in [4] and [5], where there are explicitly described the integral based loop space homology rings for the full flag manifolds as well as some generalised symmetric spaces with toral stationary subgroup.

2. Rational homotopy theory of differential graded algebras

We start with some necessary background from the rational homotopy theory [3].

Let $\mathcal{A} = (A, d_A)$ be a commutative graded differential algebra over the real numbers. A differential graded algebra (μ, d) is called *minimal model* for $\mathcal{A}$ if

(i) there exists commutative differential graded algebra morphism $h\colon (\mu, d) \longrightarrow \mathcal{A}$ inducing an isomorphism in their cohomology algebras (such h is called quasi-isomorphism);

(ii) (μ, d) is a free commutative algebra in the sense that $\mu = \wedge V$ is an exterior algebra over graded vector space V;

(iii) differential d is decomposable meaning that for a fixed set $V = \{P_\alpha, \alpha \in I\}$ of free generators of μ for any $P_\alpha \in V$, $d(P_\alpha)$ is a polynomial in generators P_β with no linear part.

If in (iii) we omit the condition "with no linear part", (μ, d) is called just a Sullivan model for (A, d).

Two algebras are said to be *weakly equivalent* if there exists quasi-isomorphism between them. This is equivalent to say that these algebras have isomorphic minimal models. The algebra (A, d_A) is said to be *formal* if it is weakly equivalent to the algebra $(H^*(A), 0)$.

The minimal model of a smooth, simply connected manifold M is by definition the minimal model of its de Rham algebra of differential forms $\Omega_{DR}(M)$. Moreover, minimal model for M is unique up to isomorphism and completely determines its rational homotopy type. Such M is said to be formal (in the sense of Sullivan) if $\Omega_{DR}(M)$ is a formal algebra. When the manifold M is formal its minimal model coincides with the minimal model of its cohomology algebra with the differential equal to zero. Note also that the rational homotopy groups for M are determined by the graded vector space V, that is

$$\dim(\pi_k(M)\otimes\mathbb{Q}) = \dim V_k, \;\; k\geq 2,$$

where V_k is the subspace in V consisting of the elements of degree k.

It is useful for further consideration to recall that the manifold M is said to have good cohomology if

$$H^*(M,\mathbb{Q}) \cong \mathbb{Q}[u_1,\dots,u_n]/\langle P_1,\dots,P_k\rangle,$$

where the polynomials $P_1,\dots,P_k \in \mathbb{Q}[u_1,\dots,u_n]$ form a regular sequence in $\mathbb{Q}[u_1,\dots,u_n]$ or equivalently $\langle P_1,\dots,P_k\rangle$ is a Borel ideal in $\mathbb{Q}[u_1,\dots,u_n]$, where $\deg u_i \geq 2$ for $1\leq i\leq n$. Such manifold is known to be formal and its minimal model is, according to [2], given by

$$\mu(M) = \mathbb{Q}[u_1,\dots,u_n]\otimes\wedge(v_1,\dots,v_k),\; du_i = 0,\; dv_j = P_j, \tag{1}$$

where $\deg v_j = \deg P_j - 1$, $1\leq j\leq k$.

In particular, it follows that a simply connected manifold having free cohomology algebra is formal and its minimal model coincides with its cohomology algebra.

2.1. **The based loop space rational homology.**

Let $\mu = (\wedge V, d)$ be a minimal model of a simply connected topological space X with the rational homology of finite type. Then the differential d can be decomposed as $d = d_1 + d_2 + \dots$, where $d_i : V \to \wedge^{\geq i+1}V$ is the part of the differential d of the length $i+1$. It particular the differential d_1 is called the *quadratic part* of the differential d.

The homotopy Lie algebra $\mathfrak{L}$ for the minimal model μ is defined as follows. The underlying vector space is a graded vector space L given by

$$sL = \mathrm{Hom}(V,\mathbb{Q}),$$

where sL denotes the suspension defined by $(sL)_i = L_{i-1}$. It is defined the pairing $\langle,\rangle : V\times sL \to \mathbb{Q}$

$$\langle v, sx\rangle = (-1)^{\deg v} sx(v),$$

which can be extended to $(k+1)$-linear map $\wedge^k V\times sL\times\cdots sL \to \mathbb{Q}$ by letting

$$\langle v_1\wedge\cdots\wedge v_k; sx_k,\dots,sx_1\rangle = \sum_{\sigma\in S_k} \varepsilon_\sigma \langle v_{\sigma(1)}; sx_1\rangle\cdots\langle v_{\sigma(k)}; sx_k\rangle,$$

where S_k is the symmetric group and $\varepsilon_\sigma = \pm 1$ are determined by

$$v_{\sigma(1)}\wedge\cdots\wedge v_{\sigma(k)} = \varepsilon_\sigma v_1\wedge\cdots\wedge v_k.$$

The space L inherits the Lie brackets $[,] : L\times L \to L$ from d_1, which are uniquely defined by

$$\langle v; s[x,y]\rangle = (-1)^{\deg y+1}\langle d_1 v; sx, sy\rangle \;\text{ for } x,y\in L, v\in V. \tag{2}$$

On the other hand, on the graded vector space of the rational homotopy groups $\pi_*(\Omega M)\otimes\mathbb{Q}$ for M, can be defined the commutator by the Samelson product. The Samelson product is defined in the category of topological spaces and continuous maps by the composite

$$S^p\wedge S^q \xrightarrow{f\wedge g} \Omega M\wedge\Omega M \xrightarrow{c} \Omega M,$$

where c is given by the multiplicative commutator $c(x,y) = x\cdot y\cdot x^{-1}\cdot y^{-1}$ and $f: S^p\to\Omega M$, $g: S^q\to\Omega M$. The graded Lie algebra $L_M = (\pi_*(\Omega M)\otimes\mathbb{Q};[,])$ for which the commutator is defined by the Samelson product is called the *rational homotopy Lie algebra* for M. The following holds [3]:

Theorem 2.1. *There is an isomorphism between the rational homotopy Lie algebra L_M and the homotopy Lie algebra $\mathfrak{L}$ of the minimal model μ for M.*

Milnor and Moore showed [7] that for a path connected homotopy associative H-space X with unit, there is an isomorphism of Hopf algebras $U(\pi_*(X)\otimes\mathbb{Q})\cong H_*(X;\mathbb{Q})$, where $U(\pi_*(X)\otimes\mathbb{Q})$ is the universal enveloping algebra of the graded Lie algebra $(\pi_*(X)\otimes\mathbb{Q},[,])$. Since the loop multiplication is homotopy associative with unit, applying Milnor-Moore theorem to $X=\Omega M$ it follows that

$$H_*(\Omega M;\mathbb{Q})\cong UL_M\cong U\mathfrak{L}. \tag{3}$$

Further on,

$$U\mathfrak{L} = T(L)/\langle xy-(-1)^{\deg x\deg y}yx-[x,y]\rangle. \tag{4}$$

Example 2.1. The rational based loop space homology ring of a simply connected manifold M having free cohomology algebra, that is $H^*(M,\mathbb{Q}) = \wedge(z_1,\dots,z_n)$, is given by

$$H_*(\Omega M,\mathbb{Q}) = \mathbb{Q}[x_1,\dots,x_n],\ \ \deg x_i = \deg z_i - 1.$$

This follows immediately from (4) as the minimal model for M coincides with $(H^*(M,\mathbb{Q}), d=0)$.

3. Rational homotopy theory of compact homogeneous spaces

Let G/H be a homogeneous space, where G is a compact, connected Lie group and H its closed connected subgroup. By the Hopf theorem [1] it holds $H^*(G,\mathbb{Q}) = \wedge(z_1,\dots,z_n)$, where $\deg z_i = 2k_i-1$ and k_i are the exponents [8] for the group G, and $n=\operatorname{rk} G$. The elements z_i are known to be the universal transgressive elements in the universal fiber bundle $BG\to EG\to G$, where BG is the classifying space for G. It is known [1] that $H^*(BG,\mathbb{Q})$ is an algebra of polynomials which are invariant under the action of the Weyl group W_G for G. More precisely, if $\mathfrak{t}$ is the maximal abelian subalgebra for G then $H^*(BG,\mathbb{Q})\cong\mathbb{Q}[\mathfrak{t}]^{W_G} = \mathbb{Q}[P_1,\dots,P_n]$, where P_i corresponds to z_i by the transgression in the universal fibre bundle $BG\to EG\to G$, see [1]. Let $\mathfrak{s}$ be the maximal abelian subalgebra for H. Then $\mathfrak{s}\subset\mathfrak{t}$ and the restriction of polynomials from $\mathfrak{t}$ to $\mathfrak{s}$ induces the homomorphism $\rho^* : \mathbb{Q}[\mathfrak{t}]^{W_G} \to \mathbb{Q}[\mathfrak{s}]^{W_H}$.

The Cartan algebra (C,d) for G/H over $\mathbb{Q}$ is defined by

$$C = H^*(BH,\mathbb{Q}) \otimes H^*(G,\mathbb{Q}),\ \ d(b\otimes 1) = 0,\ d(1\otimes z_i) = \rho^*(P_i)\otimes 1.$$

Then the well known Cartan theorem [1] states that $H^*(G/H,\mathbb{Q}) \cong H^*(C,d)$. The homogeneous space is said to be of Cartan pair if one can choose n algebraically independent generators $P_1,\dots,P_n$ for $\mathbb{Q}[\mathfrak{t}]^{W_G}$ such that $\rho^*(P_{r+1}),\dots,\rho^*(P_n)$ belong to the ideal in $\mathbb{Q}[\mathfrak{s}]^{W_H}$ generated by $\rho^*(P_1),\dots,\rho^*(P_r)$, where $r = \operatorname{rk} H$. The cohomology ring of the Cartan pair homogeneous space is given by

$$H^*(G/H,\mathbb{Q}) \cong H^*(BH,\mathbb{Q})/\langle \rho^*(P_1),\dots,\rho^*(P_r)\rangle \otimes \wedge(z_{r+1},\dots,z_n). \tag{5}$$

The notion of the Cartan pair homogeneous space determines the rational formality of homogeneous spaces, that is a homogeneous space is formal if and only if it is a Cartan pair, see [8]. In particular, this implies the result which holds for all simply connected topological spaces, that any homogeneous space G/H which has a free cohomology algebra $H^*(G/H,\mathbb{Q}) = \wedge(z_1,\dots,z_k)$ is formal. Recall that a formal simply connected topological space is characterized by the condition that its minimal model coincides with the minimal model of its cohomology algebra. Therefore, it follows from (1) that the minimal model for a Cartan pair homogeneous space can be obtained by eliminating those generators for $H^*(BH,\mathbb{Q})$ which appear as a linear part in some of $\rho^*(P_1),\dots\rho^*(P_r)$. If $u_1,\dots,u_l$ are such generators for $H^*(BH,\mathbb{Q})$ and they are the linear parts of $\rho^*(P_1),\dots,\rho^*(P_l)$, then the model for G/H is given by $\mu(G/H) = (\wedge V,d)$, where

$$\begin{gathered} V = \mathcal{L}(u_{l+1},\dots,u_n,v_{l+1},\dots,v_r,z_{r+1},\dots,z_n), \\ du_i = 0,\ \ dv_j = \rho^*(P_j),\ \ dz_i = 0. \end{gathered} \tag{6}$$

4. Rational Pontrjagin homology ring of the based loop spaces on symmetric spaces

We first note that if the cohomology algebra of a homogeneous space G/H is a free algebra, $H^*(G/H,\mathbb{Q}) \cong \wedge(z_1,\dots,z_l)$, then

$$H_*(\Omega(G/H),\mathbb{Q}) = \mathbb{Q}[x_1,\dots,x_l],\ \deg x_i = \deg z_i - 1, \tag{7}$$

according to Example 2.1. For the class of generalized symmetric spaces, using the results from [11] and [12], we can say more. Recall that a homogeneous space G/H is said to be a generalized symmetric space if there exists an automorphism θ of the group G of such that $G_0^\theta \subset H \subset G^\theta$, where G^θ is a fixed point subgroup for θ and G_0^θ is an identity component for G^θ. If the automorphism θ is of finite order and the group G is semisimple and simply connected then G^θ is known to be connected [9], which implies that generalized symmetric spaces which satisfy these assumptions have the form G/G^θ. We further consider the generalised symmetric spaces of the form G/G^{θ}. Such spaces with simple compact Lie group G are classified and listed in [11]. Moreover in [11] and further in [12] for all generalized symmetric spaces G/H, it is described the embedding $\mathfrak{s} \subset \mathfrak{t}$, where $\mathfrak{s}$, $\mathfrak{t}$ are the maximal abelian subalgebras for the Lie algebras η and $\mathfrak{g}$ for H and G, respectively.

Theorem 4.1. *Let G/H be a generalized symmetric space of a simple Lie group G with $\mathrm{rk}\, G = n$, $\mathrm{rk}\, H = r$ and let $\mathfrak{g}$, $\mathfrak{h}$ be the Lie algebras for G and H, and $\xi(\mathfrak{h})$ the center for $\mathfrak{h}$. If $\mathfrak{h} = \xi(\mathfrak{h}) \oplus \mathfrak{h}'$, where $\mathfrak{h}'$ is a simple Lie algebra then the homotopy Lie algebra L for G/H is abelian and it is given by*

$$L = \mathcal{L}(v_1, \dots, v_{n-r+1}),\ \deg v_i = 2k_i - 2,$$

where k_i are the exponents for G which are not exponents for H. Moreover.

$$H_*(\Omega(G/H), \mathbb{Q}) = \mathbb{Q}[L]. \tag{8}$$

Proof. Let $x_1, \dots, x_n$ be a canonical coordinates for G on $\mathfrak{t}$ and $y_1, \dots, y_r$ the canonical coordinates for H on the maximal abelian algebra $\mathfrak{s}$ for $\mathfrak{h}'$. Recall that the canonical coordinates on $\mathfrak{t}$ are the one-forms through which the roots for G express in the standard form. In the case of the generalised symmetric spaces which satisfy the formulated condition, the canonical coordinates x_i, when restrict to $\mathfrak{s}$ maps, according to [12], as follows: $x_{i_1} \to y_1, \dots, x_{i_r} \to y_r$, while $x_i \to 0$, for $i \neq i_1, \dots, i_r$. Since W_H is a subgroup of W_G, it implies that the map $\rho^* : \mathbb{Q}[\mathfrak{t}]^{W_G} \to \mathbb{Q}[\mathfrak{s}]^{W_H}$ is surjective. Therefore, $H^*(G/H, \mathbb{Q}) = \wedge(z_1, \dots, z_{n-r+1})$, $\deg z_i = 2k_i - 1$ and the statement follows as in (7). $\square$

Theorem 4.2. *Let G/H be a generalized symmetric space of a simple compact Lie group and let k be an arbitrary exponent for G, which is not exponent for H. If $k \neq l_i + l_j$ for all exponents l_i, l_j for H which are not, counting together with their multiplicity, exponents for G, then*

$$H_*(\Omega M, \mathbb{Q}) = \mathbb{Q}[L],$$

where L is the homotopy Lie algebra for G/H.

Proof. We first note if k is a common exponent for G and H, where G/H is a generalised symmetric space, then the restriction $\rho^*(P_k)$ of the Weyl invariant generator P_k for $H^*(BG, \mathbb{Q})$ contains some Weyl invariant generator for $H^*(BH, \mathbb{Q})$ of degree k, see [12]. It follows that, when computing $H^*(G/H, \mathbb{Q})$, each generator in $H^*(G, \mathbb{Q})$ of degree $2k-1$ eliminates the generator of degree $2k$ in $H^*(BH, \mathbb{Q})$ and this elimination is injective. We deduce from (5) that the cohomology algebra for G/H does note have a generator of degree $2k-1$. Thus, the degrees of the odd degree generators for $H^*(G/H, \mathbb{Q})$ are, according to (6), given by $2k-1$, where k runs through the exponents for G which are not the exponents for H. By the condition of the statement, for any such k it holds that $k \neq l_i + l_j$, for any two exponents l_i, l_j of H, which together with their multiplicity are not the exponents for G. It implies that the differential $\rho^*(P_k)$ has no quadratic part. Therefore, the homotopy

Lie algebra for G/H is by (2) commutative, what together with (4) implies the statement. $\square$

We further consider the irreducible simply connected compact symmetric spaces. These spaces are classified by E. Cartan and their rational homotopy groups are computed in [12]. We provide the explicit description of the loop space rational Pontrjagin homology ring for some irreducible simply connected compact symmetric spaces G/H with a simple classical Lie group G. These computations illustrate the approach which analogously can be applied to all other such spaces. In the computation that follows we use the results from [11] and [12] on the infinitesimal embedding of H into G as well as the results on their rational homotopy groups.

Proposition 4.1. *The homotopy Lie algebras of the symmetric spaces $SU(2n+1)/SO(2n+1)$, $SO(2n)/Sp(n)$ and $SO(2n+1)/SO(2)\times SO(2n-1)$ are abelian and their based loop space rational Pontrjagin homology rings are as follows:*

$$H_*(\Omega(SU(2n+1)/SO(2n+1),\mathbb{Q}) = \mathbb{Q}[x_4,x_8,\dots,x_{4n}], \tag{9}$$

$$H_*(\Omega(SO(2n)/Sp(n),\mathbb{Q}) = \mathbb{Q}[x_4,x_8,\dots,x_{2n-4}], \tag{10}$$

$$H_*(\Omega(SO(2n+1)/SO(2)\times SO(2n-1)),\mathbb{Q}) \cong \wedge(z_1)\otimes\mathbb{Q}[x_{4n-2}], \tag{11}$$

where $\deg z_1 = 1$ *and* $\deg x_i = i$ *for all* i*.*

Proof. To prove (9) we use the result from [12] that all nontrivial rational homotopy groups for $SU(2n+1)/SO(2n+1)$ are of odd degrees $5,9,\dots,4n+1$, which implies that $V = \mathcal{L}(v_5,v_9,\dots,v_{4n+1})$ and $d_1 = 0$. Therefore in the Lie algebra $\mathfrak{L}$ it holds $[x,y] = 0$ for all $x,y\in L$, what together with (4) gives the formula (9).

For the formula (10), it follows from the result of [12] on the ranks of the homotopy groups for $SO(2n)/Sp(n)$ that $V = \mathcal{L}(v_5,v_9,\dots,v_{2n-3})$ and $d_1v_i = 0$, $i = 5,9,\dots 2n-3$. Therefore, the homotopy Lie algebra $\mathfrak{L}$ is commutative and (4) directly gives the formula (10).

The ranks of the homotopy groups for the space $SO(2n+1)/SO(2)\times SO(2n-1)$ are by [12] equal to 2 and $4n-1$. It follows that $V = \mathcal{L}(v_2,v_{4n-1})$ and it is obvious that $d_1v_2 = d_1v_{4n-1} = 0$. Therefore, $L = \mathcal{L}(z_1,x_{4n-2})$ and $\mathfrak{L}$ is an abelian Lie algebra, what proves formula (11). $\square$

Theorem 4.3. *The based loop space rational Pontrjagin homology ring of the symmetric space $SU(2n)/SO(2n)$ is given by*

$$H_*(\Omega(SU(2n)/SO(2n)),\mathbb{Q}) = T(z_{2n-1})\otimes\mathbb{Q}[x_4,x_6,\dots,x_{4n-4}], \tag{12}$$

where $\deg z_{2n-1} = 2n-1$ *and* $\deg x_i = i$*.*

Proof. It follows from [12] that $V = \mathcal{L}(v_{2n},v_5,\dots,v_{4n-3},v_{4n-1})$ and $d_1v_{2n} = d_1v_5 = \dots = d_1v_{4n-3} = 0$, while $d_1v_{4n-1} = v_{2n}^2$. Therefore, the corresponding Lie algebra is given by $L = \mathcal{L}(z_{2n-1},x_4,\dots,x_{4n-4},x_{4n-2})$ and the commutators we compute using formula (2):

$$\langle v_j, s[z_{2n-1},x_i]\rangle = \langle d_1v_j; sz_{2n-1},sx_i\rangle = 0,$$

since $d_1 v_j = 0$ or $d_1 v_j = v_{2n}^2$. It follows that $[z_{2n-1}, x_i] = 0$, $i = 4, 6, \dots, 4n-2$. By the same argument we conclude that $[x_i, x_j] = 0$ for $i, j = 4, \dots, 4n-2$. For z_{2n-1} we obtain that

$$\langle v_{4n-1}; s[z_{2n-1}, z_{2n-1}]\rangle = \langle d_1 v_{4n-1}; sz_{2n-1}, sz_{2n-1}\rangle =$$
$$= \langle v_{2n}^2; sz_{2n-1}, sz_{2n-1}\rangle = 2\langle v_{2n}; sz_{2n-1}\rangle\langle v_{2n}; sz_{2n-1}\rangle = 2.$$

Thus,

$$[z_{2n-1}, z_{2n-1}] = 2x_{4n-2},$$

while all other commutators are zero. Therefore, the ideal I in (4) eliminates generator x_{4n-2}, while all other elements commute. The formula (12) follows then directly from (4). □

Theorem 4.4. *The based loop space rational Pontrjagin homology ring of the symmetric space $M = U(n+1)/U(k) \times U(n-k+1)$, $n \geq 2k-1$, $k \geq 2$ is given by*

$$H_*(\Omega M, \mathbb{Q}) = T(z_1, \dots, z_{2k-1}, x_{2n-2k+2}, x_{2n-2k+4}, \dots, x_{2n}) \tag{13}$$

quotiented by the relations

$$x_i x_j = x_j x_i, \;\; z_{2l-1} x_j = x_j z_{2l-1},$$

$$z_{2l-1} z_{2m-1} + z_{2m-1} z_{2l-1} = 0, \textit{ for } l+m \neq s, \textit{ where } n-k+2 \leq s \leq n+1,$$

and

$$z_{2l-1} z_{2m-1} + z_{2m-1} z_{2l-1} = x_{2l+2m-2}, \textit{ where } n-k+2 \leq l+m \leq n+1,$$

where $\deg z_{2l-1} = 2l-1$ *and* $\deg x_j = j$.

Remark 4.1. Note that for $k = 1$ we have that M is the complex projective space $\mathbb{C}P^n$, whose based loop space rational homology ring is known. More precisely, $H_*(\Omega S^2, \mathbb{Q}) = \mathbb{Q}[x_1]$, where $\deg x_1 = 1$ and $H_*(\Omega \mathbb{C}P^n, \mathbb{Q}) = \wedge(z_{2n-1}) \otimes \mathbb{Q}[x_1]$ for $n \geq 2$, where $\deg z_{2n-1} = 2n-1$, $\deg x_1 = 1$.

Proof. In terms of the Lie algebras the space M writes as $A_n / \mathfrak{t}^1 \otimes A_{k-1} \otimes A_{n-k}$. It follows from [12] that $V = \mathcal{L}(v_2, v_4, \dots, v_{2k}, v_{2n-2k+3}, v_{2n-2k+5}, \dots, v_{2n+1})$ and using (6) we deduce that $dv_{2i} = 0$ and $dv_{2j+1} \in \mathbb{Q}[v_2, \dots, v_{2k}]$. It implies that

$$L = \mathcal{L}(z_1, z_3, \dots, z_{2k-1}, x_{2n-2k+2}, x_{2n-2k+4}, \dots, x_{2n}).$$

Therefore, by (2) we obtain that

$$[z_i, x_j] = 0 \;\text{ and }\; [x_i, x_j] = 0.$$

In order to determine the other Lie brackets in L we fist describe the differential d_1 on the generators v_{2j+1} for V. For the clearness of the exposition we present it in more detail. We follow the notation from [11] and [12]. Let y be the canonical coordinate on $\mathfrak{t}^1$ and let $w_1, \dots, w_{n+1}$ be the canonical coordinates for A_n, and $y_1, \dots, y_k$ and $y'_1, \dots, y'_{n-k+1}$ the canonical coordinates for A_{k-1} and A_{n-k}, respectively. If $H_1, \dots, H_n$ is the Chevalley basis [9] for the maximal abelian subalgebra in A_n, then $H_1,\dots,H_{k-1}$ and $H_{k+1},\dots,H_n$ give the basis for the maximal abelian subalgebras for A_{k-1} and A_{n-k}, respectively. Using Theorem 12 from [11] we calculate that the generating vector for $\mathfrak{t}^1$ is $H = H_k + \sum\limits_{j=1}^{k-1}\frac{j}{k}H_j + \sum\limits_{j=k+1}^{n}\frac{n-j+1}{n-k+1}H_j$. Using the results of paragraph 4.4 from [12] we describe the expression of the canonical coordinates on A_n in terms of the canonical coordinates on $\mathfrak{t}^1\oplus A_{k-1}\oplus A_{n-k}$ as follows:

$$w_i \to y_i + \frac{1}{k}y,\ 1\le i\le k, \tag{14}$$
$$w_i \to y_i' - \frac{1}{n-k+1}y,\ k+1\le i\le n+1.$$

The generators $v_2,\dots,v_{2k}$ correspond to the generators of $H^*(BU(k-1),\mathbb{Q})$, which are the polynomials on the maximal abelian subalgebra for $U(k-1)$ invariant under the action of the Weyl group $W_{U(k-1)}$, which is the symmetric group S_k. Thus, v_{2i} corresponds to the symmetric polynomial $\sum\limits_{l=1}^{k} y_l^i$. The differential $d(v_{2q-1}) = \rho^*(P_q)$, where P_q is the Weyl invariant generator for $H^*(BU(n),\mathbb{Q})$, that is $P_q = \sum\limits_{i=1}^{n+1} w_i^q$ and ρ^* denotes its restriction to the maximal abelian subalgebra for $\mathfrak{t}^1\oplus A_{k-1}\oplus A_{n-k}$. It follows from (14) that

$$\rho^*(P_q) = \sum_{i=1}^{k}(y_i+\frac{1}{k}y)^q + \sum_{i=k+1}^{n+1}(y_i'-\frac{1}{n-k+1}y)^q =$$
$$= \sum_{i=1}^{k} y_i^q + \sum_{i=k+1}^{n+1}(y_i')^q + \frac{q}{k}y\sum_{i=1}^{k}y_i^{q-1} - \frac{q}{n-k+1}y\sum_{i=k+1}^{n+1}(y_i')^{q-1} + Q_q(y_i,y_i',y).$$

Note that $q\ge 3$ as $k\ge 2$. It implies that each summand of the polynomial $Q_q(y_i,y_i',y)$ contains y^2, which further gives that d_1v_{2q-1} is contained in

$$\sum_{i=1}^{k} y_i^q + \sum_{i=k+1}^{n+1}(y_i')^q + \frac{q}{k}y\sum_{i=1}^{k}y_i^{q-1} - \frac{q}{n-k+1}y\sum_{i=k+1}^{n+1}(y_i')^{q-1}.$$

We further use the fact that the power sum symmetric polynomial $\sum\limits_{i=1}^{n} x_i^j$, where $j\ge n+1$ expresses through the power sum symmetric polynomials $\sum\limits_{i=1}^{n} x_i^l$, $1\le l\le n$ such that each of them contributes in the expression.

Therefore, if $q = 2s-1$, where $n-k+2\le s\le n+1$ and $s\ne l+m$ for $1\le l$, $m\le k$, we have that $d_1v_q = 0$. Otherwise d_1v_q contains all products of the form $v_{2l}v_{2m}$ such that $s = l+m$. Therefore, we obtain that

$$[z_{2l-1}, z_{2m-1}] = \pm x_{2l+2m-2},\ \text{if } l+m = s,\ \text{for } n-k+2\le s\le n+1. \tag{15}$$

Otherwise, $[z_{2l-1}, z_{2m-1}] = 0$. Note that by changing the sign of the generators we can achieve the sign in (15) to be $+$. Altogether, we obtain that in the universal enveloping algebra $U\mathfrak{L}$ the following relations hold

$$x_i x_j = x_j x_i,\ z_{2l-1}x_j = x_j z_{2l-1}, \tag{16}$$

$$z_{2l-1}z_{2m-1} + z_{2m-1}z_{2l-1} = 0, \text{ for } l+m \neq s, \text{ where } n-k+2 \leq s \leq n+1, \tag{17}$$

$$z_{2l-1}z_{2m-1} + z_{2m-1}z_{2l-1} = x_{2l+2m-2}, \text{ where } n-k+2 \leq l+m \leq n+1. \tag{18}$$

Then the statement follows from (4). □

Example 1. Let $n = 5$ and $k = 3$, that is $M = U(6)/U(3) \times U(3)$. Then $L = \mathcal{L}(z_1, z_3, z_5, x_6, x_8, x_{10})$ and the non-trivial Lie brackets are by (15) given by $[z_3, z_5] = \pm x_8$, $[z_1, z_5] = \pm x_6$, $[z_3, z_3] = \pm 2x_6$ and $[z_5, z_5] = \pm x_{10}$. It implies that

$$H_*(\Omega M, \mathbb{Q}) = T(z_1, z_3, z_5)/\langle z_1^2 = z_1 z_3 + z_3 z_1 = 0, 2z_3^2 = z_1 z_5 + z_5 z_1\rangle.$$

Theorem 4.5. *The based loop space Pontrjagin rational homology ring of the symmetric space $M = SO(2n+1)/SO(2k) \times SO(2n+1-2k)$ is given by*

$$H_*(\Omega M, \mathbb{Q}) = T(z_{2k-1}, z_3, \ldots, z_{4k-5}, x_{4n-4k+2}, \ldots, x_{4n-2}), \tag{19}$$

quotiented by the relations

$$x_i x_j = x_j x_i,\ \ x_i z_l = z_l x_i,$$
$$\pm x_{i+j} = z_i z_j + z_j z_i \ \ \textit{for}\ \ i+j+1 = 4n-4k+3, \ldots, 4n-1,$$
$$z_i^2 = x_{2i} \ \ \textit{for}\ \ 2i+1 = 4n-4k+3, \ldots, 4n-1.$$

Proof. The rational homotopy groups for M are listed in [12] and using that result we have $V = \mathcal{L}(v_{2k}, v_4, v_8, \ldots, v_{4k-4}, v_{4n-4k+3}, \ldots, v_{4n-1})$. The quadratic part of the differential in the minimal model when applied to the generators $v_{2k}, v_4, \ldots, v_{4k-4}$ is trivial, that is $d_1 v_{2k} = d_1 v_4 = \ldots = d_1 v_{4k-4} = 0$. Further, $d_1 v_q = 0$ for $q = 4n-4k+3, \ldots, 4n-1$ and $q+1 \neq p_1 + p_2$ where $p_1, p_2 = 2k, 4, \ldots, 4k-4$. Otherwise $d_1 v_q$ contains all products of the form $v_{p_1} v_{p_2}$ such that $q+1 = p_1 + p_2$. It gives that $L = \mathcal{L}(z_{2k-1}, z_3, z_7, \ldots, z_{4k-3}, x_{4n-4k+2}, \ldots, x_{4n-2})$ and the Lie brackets are given as follows:

$$[x_i, x_j] = [z_l, x_j] = 0,$$
$$[z_i, z_j] = 0,\ \text{ if }\ i+j+1 \neq q = 4n-4k+3, \ldots, 4n-1.$$

For $i+j+1 = q = 4n-4k+3, \ldots, 4n-1$, $i \neq j$, by applying (2) we obtain

$$\begin{aligned}
\langle v_q; s[z_i, z_j]\rangle &= (-1)^{\deg z_j + 1}\langle d_1 v_q; sz_i, sz_j\rangle = \langle \sum_{l+m=q+1} v_l v_m; sz_i, sz_j\rangle \\
&= \sum_{l+m=q+1, l<m} \big(\langle v_l, sz_i\rangle\langle v_m, sz_j\rangle - \langle v_m, sz_i\rangle\langle v_l, sz_j\rangle\big) \\
&= \sum_{l+m=q+1, l<m} \big((-1)^{\deg v_l + \deg v_m}(sz_i(v_l)sz_j(v_m) = sz_i(v_m)sz_j(v_l)\big) \\
&= \pm\, sz_i(v_{i+1})sz_j(v_{j+1}).
\end{aligned}$$

Therefore,

$$\langle v_q; s[z_i, z_j]\rangle = \pm 1,\ i \neq j, i+j+1 = q,\ \ \langle v_q; s[z_i, z_i]\rangle = 2,\ 2i+1 = q,$$

where $q = 4n-4k+3, \ldots, 4n-1$. It implies that

$$[z_i, z_j] = \pm x_{i+j},\ \ i \neq j, i+j+1 = q,$$
$$[z_i, z_i] = 2x_{2i},\ \ 2i+1 = q,$$

and $q = 4n-4k+3, \ldots, 4n-1$. Then for such z_i, z_j in $H_*(\Omega M, \mathbb{Q}) \cong U\mathfrak{L}$ it holds $z_i z_j + z_j z_i = \pm x_{i+j}$ and $z_i^2 = x_{2i}$. The statement follows from (4). $\square$

Example 2. For $n = 5$ and $k = 2$ we have the space $M = SO(11)/SO(4) \times SO(7)$ and from the previous computation it follows

$$H_*(\Omega M, \mathbb{Q}) = \wedge(z_3^1, z_3^2) \otimes \mathbb{Q}[x_{14}, x_{18}].$$

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